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Optimal Piecewise Linear Basis Functions in Two Dimensions

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Optimal Piecewise Linear Basis Functions in Two Dimensions*

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Abstract

We use a variational approach to optimize the center point coefficients associated with the piecewise linear basis functions introduced by Stone and Adams [1], for polygonal zones in two Cartesian dimensions. Our strategy provides optimal center point coefficients, as a function of the location of the center point, by minimizing the error induced when the basis function interpolation is used for the solution of the time independent diffusion equation within the polygonal zone. By using optimal center point coefficients, one expects to minimize the errors that occur when these basis functions are used to discretize diffusion equations, or transport equations in optically thick zones (where they approach the solution of the diffusion equation). Our optimal center point coefficients satisfy the requirements placed upon the basis functions for any location of the center point. We also find that the location of the center point can be optimized, but this requires numerical calculations. Curiously, the optimum center point location is independent of the values of the dependent variable on the corners only for quadrilaterals.

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1 Introduction

Piecewise linear basis functions were introduced by Stone and Adams [1, 2] as an alternative to the use of Wachspress's rational functions [3] when employing the finite element method to solve the transport equation on polygonal grids in two dimensions, and on hexahedral grids in three dimensions. The method was further developed by Bailey, et al., for the solution of the diffusion equation [4] and for the transport equation in RZ and XYZ geometry [5, 6]. These basis functions have been shown to be effective in the solution of diffusion and transport equations, removing the need to employ nonlinear basis functions that require numerical integration to evaluate matrix elements for the finite element method.

In this paper, we provide an optimization of these piecewise linear basis functions when used to solve the diffusion equation, or related equations, in two Cartesian dimensions. Examples of related equations are Laplace's equation and a transport equation that limits to a diffusion equation in some physical limit. Our strategy is to optimize the accuracy of the interpolation provided by the basis functions for the solution of Laplace's equation within the zone, for Dirichlet boundary conditions on the perimeter. By doing so, an important property of the equilibrium solution, zero net flux for source free problems, is preserved on the perimeter of each zone.

The scalar function, $\Phi(x, y)$, that is being interpolated inside the polygon is the electric potential in the case of Laplace's equation. For the transport or diffusion of thermally emitted radiation Φ is the black body radiation energy density, $\Phi = aT^4$, where a is the radiation constant and T is the temperature.

Stone and Adams construct their basis functions in two dimensions by adding a "center" point inside the N sided polygon and connecting it to the corners of the polygon, thereby dividing it into triangles. (The corners are the nodes or vertices of the mesh that divides the problem domain into zones.) The basis functions have two roles, first they are weight functions for evaluating the energy deposition, or the source terms, and second they provide an interpolation within the polygonal zone for a function defined at its corners.

The basis functions must sum to 1 everywhere in the polygon to be useful as weight functions. Stone and Adams also require that the basis functions be able to interpolate any linear function within the polygonal zone exactly. These two requirements establish relations between the value of the basis functions at the center point, and the location of the center point. Stone and

Adams satisfy these requirements with a simple choice for the center point coefficients, $1/N$, where N is the number of basis functions, and by using the average of the corner positions for the location of the center point.

In this paper we show that, *given* a center point location, the value of Φ at that point can be chosen so that it gives the best approximation to the solution of the steady state diffusion equation (within the zone) with the given boundary conditions. Moreover, we give an explicit formula for the optimum value. By the principle of superposition, one naturally obtains optimized basis functions and we find that these basis functions automatically satisfy the two properties described above, for any location of the center point.

There is also an optimum position for the center point, and it appears that finding this position can only be done numerically except in special cases where symmetry can be exploited. We have found that for the case of a four sided zone, the optimum position of the center point is independent of the values of Φ at the corners, *i.e.* the position of the optimal center point is a geometric quantity, but that for five sides or more it depends as well on the values of Φ at the corners.

The optimal center point coefficients that we derive for N sided polygonal zones are easily incorporated into codes that use these basis functions, as they are simple functions of the corner and center point locations. Finding optimal center point locations is perhaps less useful, in that it is restricted to four sided zones if the values of Φ at the corners are not known, the usual case; but examining this optimization can lead to useful heuristics in choosing the location of a center point without resorting to a numerical solution.

We will not discuss optimizing the three dimensional extension of these basis functions, although we see no reason why similar considerations could not be applied in three dimensions.

The paper is organized as follows: In Section 2 we review the basis functions introduced by Stone and Adams, for polygons in two Cartesian dimensions, and the conditions that they must satisfy in order to be useful in the finite element method. In Section 3 we consider a strategy for optimizing the center point coefficients, developing optimized coefficients in a compact closed form, and demonstrate that they automatically satisfy the requirements of Stone and Adams, in addition to identifying just what is improved by using them. In Sections 4 and 5 we consider two special cases for zone shapes, finding that the optimal center point coefficients agree with those of Stone and Adams in the case of a rectangle, as one would expect. One can construct the exact solution to Laplace's equation (for our boundary

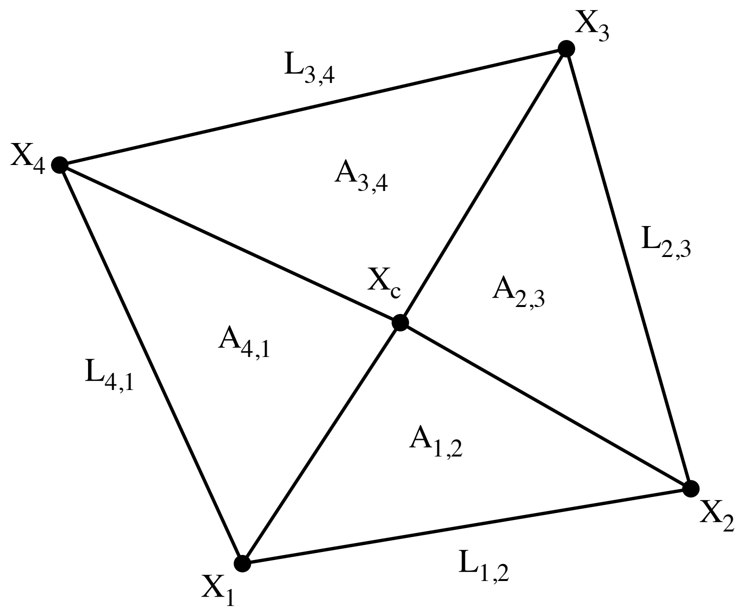

Figure 1: A quadrangle divided into four triangles. The $X_i = (x_i, y_i)$ are the locations of the corners, X_c is the location of the center point, $A_{i,j}$ is the area of the triangle, and $L_{i,j}$ is the length of an edge.

conditions) in the case of a rectangle, and we find that the basis function interpolation produces the same flux integrated along each side of the rectangle as the exact solution. In Section 6 we consider optimizing the location of the center point, providing somewhat limited, but perhaps useful, results. We end with a discussion in Section 7.

2 The piecewise linear basis functions

The basis functions are piecewise linear and continuous in a polygonal zone. In order to construct the basis functions, an N sided polygon is divided into N triangles by selecting a center point, $X_c = (x_c, y_c)$, within the polygon and connecting the center point to each corner with a straight line, as shown in Fig. 1. The location of the center point, X_c, is a free parameter subject to the requirement that it be located within the zone and that the signed areas of the triangles be positive.

There are N basis functions, χ_i, one associated with each corner of the polygonal zone located at X_i. We denote the basis function associated with corner i as χ_i. The non-zero domain of each basis function consists of N triangles; each triangle extending from the center point to a side of the polygon. The basis function associated with corner i has the value 1 at corner i, the value α_i at the center point, and the value 0 at all the other corners. The basis function is linear within each triangle, taking the form $a_{i;j,j+1}\, x + b_{i;j,j+1}\, y + c_{i;j,j+1}$ within the triangle, with the subscripts "$j, j+1$" identifying the side of the polygon; $1 \le j \le N$ and $j = N+1$ being the same as $j = 1$. The basis functions are continuous everywhere within the polygon. The coefficients $a_{i;j,j+1}$, $b_{i;j,j+1}$ and $c_{i;j,j+1}$ are unique for each triangle in each basis function. In [2], the triangular sub-cell sharing a side with the original polygon is referred to as a "side" of the polygon.

There are two requirements that the basis functions should fulfill. First, the use of the basis function as a weight function requires that

$$1 = \sum_i \chi_i(x, y) \tag{1}$$

everywhere in the zone. This requirement places a condition on the α_i,

$$1 = \sum_i \alpha_i \quad . \tag{2}$$

When Φ has a linear dependence on $X = (x, y)$ it is a solution of Laplace's equation. Therefore, a reasonable second requirement is that the basis functions be able to represent Φ exactly if its values on the boundary of the polygonal zone happen to be on a single plane in three dimensions.

Specifically, let us assume that the values, Φ_i, at the corners lie in a single plane. We have then

$$\Phi_i = A\, x_i + B\, y_i + C \quad , \tag{3}$$

where Φ_i is the value of the function at corner i and $X_i = (x_i, y_i)$ is the location of the corner. The basis functions are used to interpolate between the values provided at the corners, Φ_i. The value of $\Phi(x, y)$ anywhere in the zone is given by

$$\Phi(x, y) = \sum_i \chi_i\, \Phi_i \quad . \tag{4}$$

In particular, the value of Φ at the location of the center point, $\Phi_c = \Phi(x_c, y_c)$, is given by

$$\Phi_c = \sum_i \alpha_i\, \Phi_i \quad . \tag{5}$$

If the Φ_i satisfy Eq. (3), we have that

$$\Phi_c = \sum_i \alpha_i (A\,x_i + B\,y_i + C) \quad , \tag{6}$$

and we want

$$\Phi_c = A\,x_c + B\,y_c + C \quad . \tag{7}$$

It is easy to see that Eq. (7) will be satisfied if, in addition to Eq. (2), we have

$$x_c = \sum_i \alpha_i\, x_i \quad , \tag{8}$$

and

$$y_c = \sum_i \alpha_i\, y_i \quad . \tag{9}$$

If the interpolated value, Φ_c, lies on the same plane as the Φ_i it is easy to see that the value of Φ everywhere in the zone lies on the same plane.

Stone and Adams [1, 2] chose $\alpha_i = 1/N$ and used the average of the corner positions for the coordinates of the center point. A quick inspection of Eqs. (2,8,9) reveals that their choice satisfies these equations. This choice leads to second order accuracy [4], but restricts the polygon to shapes that contain the average of the corner positions. It also misses the possibility of tuning the basis functions for their intended use, best representing solutions of the diffusion equation (and related equations) within the zone.

3 Optimal center point coefficients

A third requirement that we can place on the basis functions is that they minimize the error when approximating the solution of the steady state diffusion equation, without sources or sinks, using a constant diffusion coefficient within the zone. Given the boundary condition of Φ_i at the corners and linear interpolation on the zone edges between them, this is the solution to Laplace's equation,

$$\nabla^2 \Phi = 0 \quad . \tag{10}$$

By providing the best approximation to the time independent diffusion equation within the zone, we hope to obtain the best accuracy that the basis functions can provide when the solution of the time dependent transport equation tends to the equilibrium diffusion limit.

A consequence of Eq. (10) is that, in steady state, the net flux around the perimeter of the polygon is zero. Mathematically, this follows from Eq. (10) by the divergence theorem. It is also required by energy conservation.

We will use the variational method for solving Laplace's equation within the polygonal zone. The method has been described many times [7, 8]; it is also called "minimizing the Dirichlet energy." In the variational method, a family of trial functions, ψ, that meet the boundary conditions are considered. The best approximation to the solution of Laplace's equation is obtained by the trial function ψ that minimizes the integral of the square of the gradient on the domain, Ω,

$$E(\psi) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 d\Omega \quad , \tag{11}$$

where $E(\psi)$ is known as the Dirichlet energy.

The piecewise linear basis function interpolation provides a parameterized trial function that matches the boundary condition on the zone with the parameters being the α_i and the location of the center point, X_c. Both α_i and X_c can be optimized to minimize the value of the integral in Eq. (11).

First, we assume that the location of the center point is given. In Section 6 we discuss its best position. Considering that Φ is a linear superposition of basis functions, Eq. (4), optimization can be accomplished one basis function at a time. As discussed above, the basis function χ_i is obtained by setting Φ_i to one, leaving the value at the center point variable, α_i, and setting the values at the other corners to 0. The gradient is constant on each triangle, thus the integral in Eq. (11) becomes a sum of the square of the gradient in each triangle multiplied by its area. This can be done by straightforward algebra. The resulting sum is quadratic in α_i. The zero of the derivative with respect to α_i provides a linear equation to solve, for each α_i independently. The resulting α_i are given by

$$\alpha_i = n_i / d \quad , \tag{12}$$

where

$$\begin{aligned} n_i = & \frac{(x_c - x_{i+1})(x_i - x_{i+1}) + (y_c - y_{i+1})(y_i - y_{i+1})}{A_{i,i+1}} \\ & + \frac{(x_c - x_{i-1})(x_i - x_{i-1}) + (y_c - y_{i-1})(y_i - y_{i-1})}{A_{i-1,i}} \end{aligned} \tag{13}$$

and

$$d = \sum_i \frac{L_{i,i+1}^2}{A_{i,i+1}} \quad , \tag{14}$$

where $L_{i,i+1}$ is the length of the edge of the polygon connecting corners i and $i+1$, and $A_{i,i+1}$ is the area of the triangle with this edge as one of its sides.

We must check that these values of α_i satisfy the requirements embodied in Eqs. (2,8,9). First, we consider the sum of the α_i. Recognizing that the α_i all share a common denominator, d, we consider the sum

$$\begin{aligned} \sum_i n_i &= \sum_i \frac{(x_c - x_{i+1})(x_i - x_{i+1}) + (y_c - y_{i+1})(y_i - y_{i+1})}{A_{i,i+1}} \\ &+ \sum_i \frac{(x_c - x_{i-1})(x_i - x_{i-1}) + (y_c - y_{i-1})(y_i - y_{i-1})}{A_{i-1,i}} \quad . \end{aligned} \tag{15}$$

Noting that the index arithmetic is modulo N and that the sum is over the entire range of indices, we shift the index i in the second sum by 1.

$$\begin{aligned} \sum_i n_i &= \sum_i \frac{(x_c - x_{i+1})(x_i - x_{i+1}) + (y_c - y_{i+1})(y_i - y_{i+1})}{A_{i,i+1}} \\ &+ \sum_i \frac{(x_c - x_i)(x_{i+1} - x_i) + (y_c - y_i)(y_{i+1} - y_i)}{A_{i,i+1}} \quad . \end{aligned} \tag{16}$$

At this point the sums can be combined term by term

$$\sum_i n_i = \sum_i \frac{L_{i,i+1}^2}{A_{i,i+1}} \quad , \tag{17}$$

and we see that

$$\sum_i n_i = d \quad , \tag{18}$$

and therefore

$$1 = \sum_i \alpha_i \quad . \tag{19}$$

This is true regardless of the location that we choose for the center point as long as the individual triangle areas are positive.

Next, we consider

$$\sum_i n_i x_i \quad . \tag{20}$$

Referring to Eq. (15), we have

$$\begin{aligned}\sum_i n_i\, x_i \;&=\; \sum_i \frac{(x_c - x_{i+1})(x_i - x_{i+1})x_i + (y_c - y_{i+1})(y_i - y_{i+1})x_i}{A_{i,i+1}} \\ &+\; \sum_i \frac{(x_c - x_{i-1})(x_i - x_{i-1})x_i + (y_c - y_{i-1})(y_i - y_{i-1})x_i}{A_{i-1,i}} \quad . \end{aligned} \tag{21}$$

As earlier, we shift the i by 1 in the second sum

$$\begin{aligned}\sum_i n_i x_i \;&=\; \sum_i \frac{(x_c - x_{i+1})(x_i - x_{i+1})x_i + (y_c - y_{i+1})(y_i - y_{i+1})x_i}{A_{i,i+1}} \\ &+\; \sum_i \frac{(x_c - x_i)(x_{i+1} - x_i)x_{i+1} + (y_c - y_i)(y_{i+1} - y_i)x_{i+1}}{A_{i,i+1}} . \end{aligned} \tag{22}$$

Reorganizing, we get

$$\begin{aligned}\sum_i n_i x_i \;&=\; \sum_i \frac{(x_c - x_{i+1})(x_i - x_{i+1})x_i + (x_c - x_i)(x_{i+1} - x_i)x_{i+1}}{A_{i,i+1}} \\ &+\; \sum_i \frac{(y_c - y_{i+1})(y_i - y_{i+1})x_i + (y_c - y_i)(y_{i+1} - y_i)x_{i+1}}{A_{i,i+1}} \,. \end{aligned} \tag{23}$$

We simplify the numerator in the first sum and factor the numerator in the second sum, revealing

$$\begin{aligned}\sum_i n_i x_i \;&=\; \sum_i \frac{(x_i - x_{i+1})^2 x_c}{A_{i,i+1}} \\ &+\; \sum_i \frac{(y_c(x_i - x_{i+1}) + x_{i+1}y_i - x_i y_{i+1})(y_i - y_{i+1})}{A_{i,i+1}} \quad . \end{aligned} \tag{24}$$

To sort this out, we consider

$$2A_{i,i+1} = x_c(y_i - y_{i+1}) + y_c(x_{i+1} - x_i) - x_{i+1}y_i + x_i y_{i+1} \quad , \tag{25}$$

obtained with the cross product rule for the area of the triangle, followed by factoring out x_c and y_c. With an eye to Eq. (24), we manipulate this further, obtaining

$$y_c(x_i - x_{i+1}) + x_{i+1}y_i - x_i y_{i+1} = x_c(y_i - y_{i+1}) - 2A_{i,i+1} \quad , \tag{26}$$

and we substitute this into Eq. (24) to obtain

$$\begin{aligned}\sum_i n_i x_i &= \sum_i \frac{(x_i - x_{i+1})^2 x_c}{A_{i,i+1}} + \sum_i \frac{(y_i - y_{i+1})^2 x_c}{A_{i,i+1}} \\ &+ 2\sum_i (y_{i+1} - y_i) \quad .\end{aligned} \tag{27}$$

The last sum in Eq. (27) is zero, we now have

$$\sum_i n_i x_i = x_c \sum_i \frac{L_{i,i+1}^2}{A_{i,i+1}} = x_c d \quad , \tag{28}$$

and finally,

$$x_c = \sum_i \alpha_i x_i \quad . \tag{29}$$

We refrain from repeating this derivation to show that

$$y_c = \sum_i \alpha_i y_i \quad . \tag{30}$$

With this done, we have shown that the optimal α_i produce satisfactory weight functions for the finite element method and satisfy the requirement that the basis functions can represent any linear solution exactly. We did not have an a-priori guarantee that the optimal center point coefficients would lead to basis functions that satisfy the requirements above for any choice of center point that provides positive area triangles, but they do.

We have also considered how to determine α_i so that the perpendicular gradient (the flux) integrated around the perimeter of the polygonal zone is zero. This requirement leads to exactly the same α_i as we just obtained by minimizing the square of the gradient. We see that using the optimal α_i preserves an important property of the exact solution, that the divergence of the gradient is zero, or, in other terms, that energy is conserved within the zone.

4 Rectangles

The exact solution for Laplace's equation in the interior of a rectangle, for any given boundary values, Φ_i, on the corners and a linear interpolation

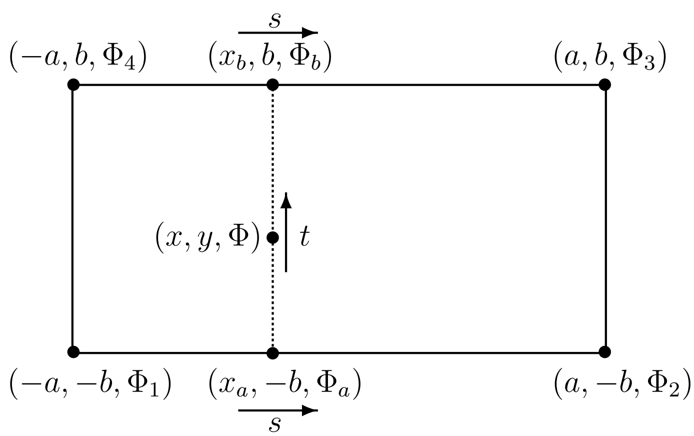

Figure 2: A rectangular zone with corner coordinates and values of Φ annotated.

between them on the edges of the rectangle, can be expressed in closed form. This provides a point of comparison for the piecewise linear basis function interpolation, both in terms of investigating the accuracy of the flux across the edges and relating the optimal center point coefficients to the ones used by Stone and Adams.

First, let us consider the exact solution for Laplace's equation on the annotated rectangle shown in Fig. 2.

We construct a "ruled surface" for Φ by defining

$$\begin{aligned} x &= -a + 2as \\ y &= -b + 2bt \\ \Phi_a &= \Phi_1 + (\Phi_2 - \Phi_1)s \\ \Phi_b &= \Phi_4 + (\Phi_3 - \Phi_4)s \\ \Phi &= \Phi_a + (\Phi_b - \Phi_a)t \quad , \end{aligned} \tag{31}$$

where s and t is are parameters that vary from 0 to 1. This parametrized definition for $\Phi(x, y)$ matches the desired boundary conditions on the rectangle, by inspection. For the rectangular domain, the ruled surface is easily expressed directly as a function of x and y,

$$\begin{aligned} \Phi(x, y) = \frac{1}{4} \Big[& (-\Phi_1 + \Phi_2 + \Phi_3 - \Phi_4)\frac{x}{a} + (-\Phi_1 - \Phi_2 + \Phi_3 + \Phi_4)\frac{y}{b} \\ & + (\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4)\frac{xy}{ab} + (\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) \Big] \quad . \end{aligned} \tag{32}$$

The fact that the Laplacian is zero is easily seen by inspecting Eq. (32) and the fact that the boundary conditions are matched is easily seen by inspecting Eq. (31). We have, then, that the ruled surface interpolation of the boundary conditions is the exact solution of Laplace's equation on the rectangle. Unfortunately, a ruled surface does not provide a solution of Laplace's equation on a more general quadrilateral.

Using the scheme of Stone and Adams the center point is the average of the corner positions, located at the origin in this case, and the $\alpha_i = 1/4$. For the same choice of the center point, the optimal α_i, from Eq. (12), are also all 1/4. The optimal α_i therefore match those used by Stone and Adams in the symmetric case of the rectangle, but an examination of Eq. (12) shows that distorted zones lead to unequal α_i. We expect that using the optimal α_i will lead to accuracy improvements for distorted zones.

5 Boomerangs

Lagrangian hydrodynamic algorithms function by moving the nodes that define the mesh. This motion can lead to a zone taking on a concave shape, the four sided version of which is known informally as a boomerang. As a particular example, consider the symmetric case shown in Fig. 3. The coordinates of the corners have been chosen so that they are symmetric about the y axis, and that the center point defined by the average of the corner positions is at the origin.

In our example boomerang, when $2b > c$ the center point defined by the average of the corner positions, the origin, is located outside the zone and the scheme of Stone and Adams breaks down. When using the optimal α_i one is free to choose the center point to be within the zone, perhaps at a sweet spot half way between corners 2 and 4, and get a perfectly good division of the boomerang into four triangles.

6 Optimal center point location

With the optimal center point coefficients calculated as a function of the location of the center point, (x_c, y_c), one can consider optimizing the location of the center point. Just as we found the optimal α_i by minimizing the square of the gradient, Eq. (11), one can attempt using the same strategy to optimize

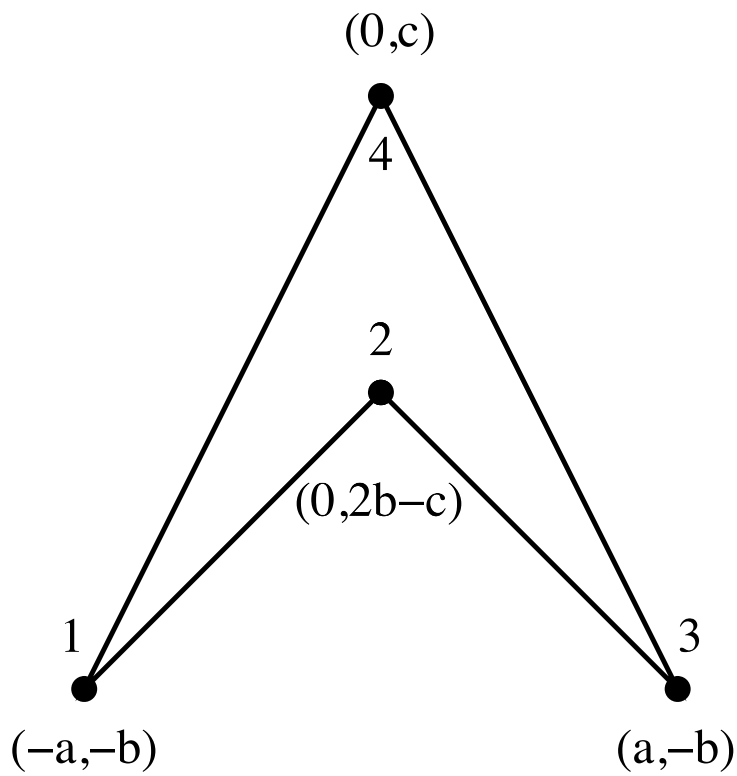

Figure 3: A rather symmetric boomerang with corner numbers and coordinates annotated.

the location of the center point. Just as we had no a-priori guarantee that the α_i calculated by the variational method would lead to satisfactory basis functions, we have no guarantee that the optimal location for the center point is independent of the values, Φ_i, on the corners.

The process for calculating the optimal center point is simply described, but not easily executed: after substituting the optimal α_i, find the location of the center point that minimizes the square of the gradient of Φ. In practice, one wants to find the zeros of the derivatives of the square of the gradient with respect to x_c and y_c. The expressions involved are complex and we have not been able to find a closed form solution for the general case, even for the four sided zone. Of course, for it to be useful, the location of the center point needs to be independent of the values of the Φ_i.

We have explored the issue numerically, however, using Mathematica to find the center point that minimizes the square of the gradient, employing high precision arithmetic. We have found that the location of the optimum center point is independent of the values for the Φ_i on four sided zones, including the boomerang case. We also find that the resulting center point is independent of the starting point in a numerical search as long as one starts from a position where all of the triangles the zone is decomposed into have positive areas. The situation is not so rosy for zones with a number of sides greater than 4, however. In all of the cases we have examined the optimal center point is different for each basis function, leading to the conclusion that it depends on the values for the Φ_i. We note that for asymmetric four sided zones, the optimal location for the center point does not correspond to the average of the locations of the corners.

For symmetric cases, further results can be obtained analytically. For the rectangle shown in Fig. 2 the integral of the square of the gradient can be evaluated in closed form. For all four basis functions it is

$$\frac{3a^2 + 3b^2 + {x_c}^2 + {y_c}^2}{2ab} \quad , \tag{33}$$

and we see that the zeros of the derivatives with respect to x_c and y_c will occur at $x_c = 0$ and $y_c = 0$. This is not a surprise, considering the symmetry of the zone.

The boomerang of Fig. 3 is a little more instructive. The integral of the square of the gradient for each basis function is a unique expression that is too large to include here, but the derivative with respect to x_c unveils an overall factor of x_c, expected from the symmetry, exposing the root at

$x_c = 0$. Setting $x_c = 0$ in the integral of the square of the gradient for the basis functions, and then taking the derivative with respect to y_c, we obtain

$$\frac{4(b-c)(a^2(b-y_c)+(b^2-c^2+2b(c-y_c))(b+y_c))}{a(a^2+(b-c)^2+4by_c)^2} \tag{34}$$

for the basis function associated with corner 1 and the same rational function of y_c, up to a multiplicative factor involving a, b and c for the other basis functions.[1] The denominator is positive and we have a zero for this expression when

$$a^2(b-y_c)+(b^2-c^2+2b(c-y_c))(b+y_c)=0 \quad . \tag{35}$$

This is a quadratic in y_c, with the solutions

$$y_c = \frac{\pm\sqrt{(a^2+(c-3b)^2)(a^2+(b+c)^2)}-a^2-(b-c)^2}{4b} \quad . \tag{36}$$

We take the + sign for the square root in order to be sure that the center point is inside the zone.

7 Discussion

The piecewise linear basis functions introduced by Stone and Adams have been shown to be an effective strategy for discretizing the diffusion equation, and transport equations that limit to the diffusion equation, using the finite element method. Stone and Adams chose the average of the corners of the polygon for the location of the center point, and $1/N$ for the value of the basis functions there, in their original definition. This produces useful basis functions and prior work has shown that the resulting discretization produces second order accuracy, but leaves open the question of further optimization.

We have developed a variational approach to optimize the values of the basis functions at the center point location, obtaining a compact closed form that is easily used in a computer program. We expect that the accuracy of the finite element discretization will improve by using the optimal center point coefficients, and, in particular, errors due to a failure to provide zero divergence of the flux on the perimeter of a zone for a source free problem, as

[1] This general theme persists for asymmetric quadrilateral zones when numerical coordinates are used for the corners.

the solution approaches equilibrium, will be removed by using these center point coefficients.

Using the same variational approach, it is possible to optimize the location of the center point, although numerical solution appears to be required in the general case. We find, numerically, that the optimal location of the center point is independent of the boundary conditions only for the case of four sides, restricting the utility of this second level of optimization to meshes composed of quadrilateral zones.

We have not explored optimizing the extension of these basis functions used for three dimensional problems, although we would expect that the strategy that we have developed would be useful in three dimensions as well.

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